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On the Dimension Formula for the Spaces of Jacobi Forms of Degree Two

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§1. Result

Let $\mathfrak{S}_g = \{Z \in M_g(\mathbf{C}) \mid {}^tZ = Z,\ \mathrm{Im}\, Z > 0\}$ be the Siegel upper half plane of degree g and let $\Gamma_g = Sp(g, \mathbf{Z})$. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, we denote $(AZ+B)(CZ+D)^{-1}$ by $M\langle Z\rangle$. Let $\mathbf{e}(z)$ denote $\exp(2\pi i z)$.

Definition 1.1. Let k, m be positive integers. A holomorphic function $f(Z,W)$ on $\mathfrak{S}_g \times \mathbf{C}^g$ is called *a Jacobi form of weight k and index m with respect to* Γ_g, if it satisfies the following transformation formulas and a regularity condition at infinity:

(1) $$f(M\langle Z\rangle, {}^t(CZ+D)^{-1}W) = \det(CZ+D)^k \mathbf{e}(m\,{}^tW(CZ+D)^{-1}CW) f(Z,W),$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$,

(2) $$f(Z, W + Z\lambda + \mu) = \mathbf{e}(-m({}^t\lambda Z\lambda + 2\,{}^t\lambda W)) f(Z,W), \quad \text{for any } \lambda, \mu \in \mathbf{Z}^g.$$

If f satisfies (1) and (2), f has a Fourier expansion of the form:

$$f(Z,W) = \sum_{N,r} c(N,r)\, \mathbf{e}(\mathrm{Tr}(NZ) + {}^tr W),$$

where N is over the symmetric half integral matrix of degree g and r is over the integral g-vector. The regularity condition at infinity is:

(3) $$c(N,r) = 0 \text{ unless } 4mN - r\,{}^tr \text{ is semi-positive.}$$

Remark 1.2. If $g \geq 2$, then the condition (3) is superfluous ([Sh], [Y] and [Z]).

Definition 1.3. A Jacobi form f is called *a Jacobi cusp form* if $c(N,r) = 0$ unless $4mN - r\,{}^tr$ is positive definite in the Fourier expansion above.

Our main result is the following

Theorem 1.4. *If $k \geq 4$, then the dimension of the space of Jacobi cusp forms $J_{k,m}^{cusp}(\Gamma_2)$ with respect to Γ_2 is given by the following Mathematica function:*

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JacobiTwo[k_,m_]:=Block[{a,lk,x,y,m2,m3,m4,r,p,S1e,S1,S2e,S2,S3,SS,SSS},
   mod[x_,y_]:=Mod[x,y]+1;
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m2=Mod[m,2];
m3=Mod[m,3];
m4=Mod[m,4];

r=0;
While[EvenQ[m/2^r],r++;];
p=m/2^r;

S1e=4*Sum[Mod[x^2,m],{x,1,m-1}];
S1=S1e+Sum[Mod[(2*x-1)^2,4*m],{x,1,m}];
S2e=16*Sum[Mod[x^2,m]^2,{x,1,m-1}];
S2=S2e+Sum[Mod[(2*x-1)^2,4*m]^2,{x,1,m}];
S3=Sum[Mod[x^2,4*m]^3,{x,1,2*m-1}];
SS=4*Sum[Mod[x^2,4*m]*Mod[x^2,m],{x,1,2*m-1}];
SSS=Sum[Mod[x^2,4*m]*Mod[y^2,4*m]*Mod[(x-y)^2,4*m],{x,1,2*m-1},{y,1,2*m-1}];

a=m^2*((2*k-3)*(2*k-4)*(2*k-5)/2^8/3^3/5-(2*k-4)/2^4/3^2+1/2^3/3);
a=a+(3*k-20)*S1/2^5/3^2+(-k+7)*S2/m/2^7/3+S3/m^2/2^8/3^2;
a=a+S1^2/m^2/2^7-S1*S2/m^3/2^8+SSS/m^3/2^8/3;

a=a+(2*k-3)*(2*k-4)*(2*k-5)/2^8/3^3/5+(2-k)/2^3/3^2+1/2^3/3;
a=a-m4^3/2^5/3^2+(10-k)*m4^2/2^7/3+(3*k-20)*m4/2^5/3^2;

lk={1,-1};
a=a+lk[[mod[k,2]]]*m*((2*k-3)*(2*k-5)/2^7/3^2-(k-2)/2^3/3+1/2^3);
a=a+lk[[mod[k,2]]]*(S1*(k-8)/m/2^6/3+S1*m4/m/2^7+m4*m*(k-8)/2^6/3);

lk={1,-1};
a=a+lk[[mod[k,2]]]*m*((2*k-3)*(2*k-5)/2^7/3+(5-3*k)/2^4/3+7/2^4/3);
a=a+lk[[mod[k,2]]]*(4*SS+(16-2*m4)*m*S1+4*(k-7+m4)*m*S1e-8*S2-S2e)/m^2/2^8;

a=a+(k-2)/2^4/3;
a=a+(m4-4)/2^5;

a=a+If[m2==0,1,0]*(k-2)/2^5;
a=a+If[m2==0,1,-1]*m4/2^5-If[m2==0,1,0]/2^3;

a=a+If[m3==0,3,1]*(k-2)/2/3^3;
a=a-If[m3==0,3,1]/2^2/3+If[m3==0,1,0]*Mod[m/3,3]/3^2;

a=a+(k-2)/2/3^3;
a=a-1/2^2/3;

lk={-1,-2*k+4,1,2*k-4};
a=a+If[m2==0,1,0]*lk[[mod[k,4]]]*m/2^5/3;
lk={0,-1,0,1};
a=a+If[m2==0,1,0]*lk[[mod[k,4]]]*(S1/m-4*m)/2^5;

lk={2*k-4,-1,-2*k+4,1};
a=a+If[m2==0,1,0]*lk[[mod[k,4]]]/2^5/3;
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lk={1,0,-1,0};
a=a+If[m2==0,1,0]*lk[[mod[k,4]]]*(m4-4)/2^5;

lk={0,-1,0,1};
a=a+lk[[mod[k,4]]]*Sum[Mod[(2*y+m2)^2,8*m]-Mod[(2*y+m2)^2+4*m,8*m],
      {y,0,m-1}]/m/2^6;

lk={1,0,-1,0};
a=a+If[m2==0,-1,0]*lk[[mod[k,4]]]/2^4;
a=a+lk[[mod[k+m,4]]]*(Mod[m,8]-Mod[m+4,8])/2^6;

If[m3==0,Goto[label1a],0];
lk={-2*k+3,-2*k+5,4*k-8};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lk[[mod[k,3]]]*m/2^4/3^3;
lk={-1,-1,2};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lk[[mod[k,3]]]*(S1/m-4*m)/2^4/3^2;
Goto[label2a];
Label[label1a];
lk={6*k-13,-6*k+11,2};
a=a+lk[[mod[k,3]]]*m/2^4/3^3;
lk={1,-1,0};
a=a+lk[[mod[k,3]]]*(S1/m-4*m)/2^4/3;
Label[label2a];

lk={2*k-3,2*k-5,-4*k+8};
a=a+lk[[mod[k,3]]]/2^4/3^3;
lk={1,1,-2};
a=a+lk[[mod[k,3]]]*(m4-4)/2^4/3^2;

lk={-2*k+3,-2*k+5,2,2*k-3,2*k-5,-2};
a=a+lk[[mod[k,6]]]*m/2^4/3^2;
lk={-1,-1,0,1,1,0};
a=a+lk[[mod[k,6]]]*(S1/m-4*m)/2^4/3;

If[m3==0,Goto[label1b],0];
lk={2*k-3,2*k-5,-2,-2*k+3,-2*k+5,2};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lk[[mod[k,6]]]/2^4/3^2;
lk={1,1,0,-1,-1,0};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lk[[mod[k,6]]]*(m4-4)/2^4/3;
Goto[label2b];
Label[label1b];
lk={2*k-7,-2*k+1,-4*k+8,-2*k+7,2*k-1,4*k-8};
a=a+lk[[mod[k,6]]]/2^4/3^2;
lk={1,-1,-2,-1,1,2};
a=a+lk[[mod[k,6]]]*(m4-4)/2^4/3;
Label[label2b];
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If[m3==0,Goto[label1c],0];
lk={-1,-1,2};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*Sum[If[Mod[y,3]==0,2,-1]*
      (lk[[mod[k,3]]]*Mod[y^2,12*m]+lk[[mod[k+1,3]]]*Mod[y^2+4*m,12*m]+
      lk[[mod[k+2,3]]]*Mod[y^2+8*m,12*m]),{y,0,2*m-1}]/m/2^4/3^3;
lk={1,-1,0};
a=a+Sum[If[Mod[y,3]==0,0,1]*(lk[[mod[k,3]]]*Mod[y^2,12*m]+lk[[mod[k+1,3]]]*
      Mod[y^2+4*m,12*m]+lk[[mod[k+2,3]]]*Mod[y^2+8*m,12*m]),{y,0,2*m-1}]/m/
      2^4/3^2;
Goto[label2c];
Label[label1c];
lk={1,-1,0};
a=a+Sum[lk[[mod[k,3]]]*Mod[9*y^2,12*m]+lk[[mod[k+1,3]]]*Mod[9*y^2+4*m,12*m]+
      lk[[mod[k+2,3]]]*Mod[9*y^2+8*m,12*m],{y,0,2*m/3-1}]/m/2^3/3^2;
Label[label2c];

lk={-1,0,1};
a=a+lk[[mod[k,3]]]/2/3^2;
lk={1,1,-2};
a=a+(lk[[mod[k+m,3]]]*Mod[m,12]+lk[[mod[k+m+1,3]]]*Mod[m+4,12]+
      lk[[mod[k+m+2,3]]]*Mod[m+8,12])/2^3/3^3;

a=a+0;

lk={0,1,0,-1};
a=a+If[m2==0,1,0]*lk[[mod[k,4]]]/2^3;

lk={-1,1,0};
a=a+lk[[mod[k,3]]]*If[m3==0,-3,1]/2^3/3^3;

lk={-1,1,0};
a=a+lk[[mod[k,3]]]/2^3/3^3;

If[m3==0,Goto[label2d],0];
lk={1,-1};
a=a+lk[[mod[k,2]]]*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],1,-1]/2^2/3^3;
Label[label2d];

If[m3==0,Goto[label1e],0];
lk={1,-1,-2,-1,1,2};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],1,-1]*lk[[mod[k,6]]]/2^2/3^2;
Goto[label2e];
Label[label1e];
lk={1,1,0,-1,-1,0};
a=a+lk[[mod[k,6]]]/2^2/3;
Label[label2e];
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If[m3==0,Goto[label1f],0];
lk={1,-1,-2,-1,1,2};
a=a+If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]*lk[[mod[k,6]]]/2^2/3^3;
Goto[label2f];
Label[label1f];
lk={1,1,0,-1,-1,0};
a=a+lk[[mod[k,6]]]/2^2/3^2;
Label[label2f];

lk={-1,1,0};
a=a+lk[[mod[k,3]]]/2^2/3;

If[m2==1,Goto[label2g],0];
If[m3==0,Goto[label1g],0];
lk={-1,-1,-2,-1,-1,0,1,1,2,1,1,0};
a=a+lk[[mod[k,12]]]*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]/2^3/3;
Goto[label2g];
Label[label1g];
lk={3,1,0,-1,-3,-2,-3,-1,0,1,3,2};
a=a+lk[[mod[k,12]]]/2^3/3;
Label[label2g];

lk={1,1,0,-1,-1,-2,-1,-1,0,1,1,2};
a=a+lk[[mod[k,12]]]*If[m2==0,1,0]/2^3/3;

If[m3==0,Goto[label2h],0];
lk={1,-1};
a=a+lk[[mod[k,2]]]*If[Mod[p,3]==1,1,-1]*If[EvenQ[r],-1,1]/2^2/3^2;
Label[label2h];

a=a+0;

If[Mod[m,5]==0,Goto[label1i],0];
lk={0,-1,0,1,0};
a=a+lk[[mod[k,5]]]*If[Mod[p^2,5]==1,1,-1]*If[EvenQ[r],-1,1]/2/5;
Goto[label2i];
Label[label1i];
lk={2,1,0,-1,-2};
a=a+lk[[mod[k,5]]]/2/5;
Label[label2i];

lk={0,1,0,-1,0};
a=a+lk[[mod[k,5]]]/2/5;

Return[a];

]
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§2. Methods

Definition 2.1. Let a and b be rational g-vectors. The theta function $\theta_{a,b}(Z,W)$ with characteristic (a,b) is a holomorhic function on $\mathfrak{S}_g \times \mathbf{C}^g$ defined by

$$\sum_{q\in\mathbf{Z}^g} \mathbf{e}((1/2)\,{}^t(q+a)Z(q+a)+{}^t(q+a)(W+b)).$$

For any integral g-vector r, we have

$$\theta_{a+r,b}(Z,W)=\theta_{a,b}(Z,W).$$

Hence $\theta_{a,0}(Z,W)$ depends only on a mod $\mathbf{Z}^g$. So we assume a is an element of $\mathbf{Q}^g/\mathbf{Z}^g$. If a runs a complete set of representatives of $\frac{1}{2m}\mathbf{Z}^g/\mathbf{Z}^g$, then $\theta_{a,0}(2mZ,2mW)$ form a basis of theta function of degree $2m$. Therefore if f is a Jacobi form of index m, there exist uniquely determined holomorphic functions $f_r(Z)$ ($r \in \frac{1}{2m}\mathbf{Z}^g/\mathbf{Z}^g$) on $\mathfrak{S}_g$ satisfying

$$f(Z,W)=\sum_r f_r(Z)\,\theta_{r,0}(2mZ,2mW).$$

We define $(2m)^g$-vectors:

$$F(Z)=(f_r(Z)),\quad \Theta(Z,W)=(\theta_{r,0}(2mZ,2mW)).$$

Then by definition we have

$$(*)\qquad {}^tF(M\langle Z\rangle)\Theta(M\langle Z\rangle,{}^t(CZ+D)^{-1}W)$$
$$=\det(CZ+D)^k\mathbf{e}(m\,{}^tW(CZ+D)^{-1}CW)\,{}^tF(Z)\Theta(Z,W).$$

We need the following transformation formula for the theta functions ([Si]).

Proposition 2.2. *Let* $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_g$. *Then for any* $r\in\frac{1}{2m}\mathbf{Z}^g/\mathbf{Z}^g$, *we have*

$$\theta_{r,0}(2mM\langle Z\rangle,2m\,{}^t(CZ+D)^{-1}W)$$
$$=\det(CZ+D)^{1/2}\mathbf{e}(m\,{}^tW(CZ+D)^{-1}CW)\times\sum_s u_{rs}(M)\,\theta_{s,0}(2mZ,2mW),$$

where s runs a complete set of representatives of $\frac{1}{2m}\mathbf{Z}^g/\mathbf{Z}^g$ and $(u_{rs}(M))_{r,s}$ is an unitary matrix of degree $(2m)^g$ depending only on M and the choice of $\det(CZ+D)^{1/2}$.

Let $u(M)=(u_{rs}(M))_{r,s}$. Then by the proposition we have

$$(**)\quad \Theta(M\langle Z\rangle,{}^t(CZ+D)^{-1}W)=\det(CZ+D)^{1/2}\mathbf{e}(m\,{}^tW(CZ+D)^{-1}CW)u(M)\Theta(Z,W).$$

From $(*)$ and $(**)$ we have

$$F(M\langle Z\rangle)=\det(CZ+D)^{k-1/2}\,\overline{u(M)}\,F(Z).$$

Namely, $F(Z)$ is a vector valued modular form with respect to the automorphic factor:

$$\det(CZ+D)^{k-1/2}\,\overline{u(M)}.$$

So we have the following proposition ([Sh]).

Proposition 2.3. *By the mapping: $f(Z,W) \mapsto F(Z)$, $J_{k,m}(\Gamma_g)$ is mapped isomorphically to the space of the vector valued modular forms with respect to the above automorphic factor and Γ_g.*

Let $\Theta(Z) = \sum_{\eta \in \mathbf{Z}^g} \mathbf{e}({}^t\eta Z \eta)$ and let

$$\Gamma_0^g(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C \equiv O \pmod 4 \right\}.$$

If $M \in \Gamma_0^g(4)$, then $J(M,Z) := \Theta(M\langle Z\rangle)/\Theta(Z)$ is holomorphic on $\mathfrak{S}_g$ and satisfies

$$J(M,Z)^2 = \det(CZ+D)\left(\frac{-1}{\det D}\right).$$

Let $\Gamma_g(N)$ be the principal congruence subgroup of level N of Γ_g. Namely,

$$\Gamma_g(N) = \{M \in \Gamma_g \mid M \equiv 1_{2g} \pmod N\}.$$

If $M \in \Gamma_g(4)$, we may assume that $\det(CZ+D)^{1/2} = J(M,Z)$. Then $u(M)$ becomes a representation of $\Gamma_g(4)$.

$\Gamma_g(N)$ is a normal subgroup of Γ_g. If $N \geq 3$, $\Gamma_g(N)$ acts on $\mathfrak{S}_g$ without fixed points and the quotient space $X_g(N) := \Gamma_g(N)\backslash\mathfrak{S}_g$ is a (non-compact) manifold. $X_g(N)$ is a open subspace of a projective variety $\overline{X}_g(N)$ which was constructed by I. Satake ([Sa], Satake compactification). If $g \geq 2$, $\overline{X}_g(N)$ has singularities along its cusps: $\overline{X}_g(N) - X_g(N)$. Cusps of $\overline{X}_g(N)$ is (as a set) a disjoint union of copies of $X_{g'}(N)$'s ($0 \leq g' < g$). A desingularization $\widetilde{X}_g(N)$ of $\overline{X}_g(N)$ was constructed by J.-I. Igusa ($g = 2$, 3, 4) ([Ig2]) and Y. Namikawa ([Nm]) and more generally by D. Mumford and others ([AMRT], Toroidal compactification).

Let $\mathcal{V}_m$ be $\mathfrak{S}_g \times \mathbf{C}^{(2m)^g}$. $\Gamma_g(4N)$ acts on $\mathcal{V}_m$ as follows:

$$M(Z,v) = (M\langle Z\rangle, \overline{u(M)}v).$$

$V_m := \Gamma_g(4N)\backslash\mathcal{V}_m$ is non-singular and is a vector bundle over $X_g(4N)$. Let $\mathcal{H}_g$ be $\mathfrak{S}_g \times \mathbf{C}$. $\Gamma_g(4N)$ acts on $\mathcal{H}_g$ as follows:

$$M(Z,v) = (M\langle Z\rangle, J(M,Z)v).$$

$H_g := \Gamma_g(4N)\backslash\mathcal{H}_g$ is a line bundle over $X_g(4N)$. H_g is extended to a line bundle $\widetilde{H}_g$ over $\widetilde{X}_g(4N)$ and also to an ample line bundle $\overline{H}_g$ over $\overline{X}_g(4N)$.

Now we have

Proposition 2.4. *If $m \mid N$, V_m is extendable to a vector bundle $\widetilde{V}_m$ over $\widetilde{X}_g(4N)$. $\widetilde{V}_m$ is a flat vector bundle and the Chern class $c_i(\widetilde{V}_m)$ $(i \geq 1)$ is 0.*

Hence in the following we assume that the level is divisible by $4m$. Let $J_{k,m}(\Gamma_g(4mN))$ be the space of Jacobi forms with respect $\Gamma_g(4mN)$. This is canonically identified with the space

$$\Gamma(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)})),$$

which is the space of the global holomorphic sections of $\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)}$. Let $D := \widetilde{X}_g(4mN) - X_g(4mN)$ be the divisor at infinity. D is a divisor with simple normal crossings. $J_{k,m}^{cusp}(\Gamma_g(4mN))$ is canonically identified with the space

$$S_{k,m}(\Gamma_g(4mN)) := \Gamma(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)} - D)).$$

$\mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)} - D)$ is the sheaf of germs of holomorphic sections which vanish along D and this is isomorphic to $\mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)} \otimes [D]^{\otimes(-1)})$, where $[D]$ is the line bundle associated with D. And this is isomorphic to $\mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-2g-3)} \otimes K_{\widetilde{X}_g(4mN)})$, since $K_{\widetilde{X}_g(4mN)} \otimes [D] \simeq \widetilde{H}_g^{\otimes(2g+2)}$.

Since $\widetilde{V}_m$ is a flat vector bundle and $\widetilde{H}_g$ is positive on $X_g(4mN)$, we can prove the following theorem by the vanishing theorem of Kodaira-Nakano ([Kd], [Nk]).

Theorem 2.5. *If $k \geq g+2$ and $p > 0$, then*

$$H^p(\widetilde{X}_g(4mN), \mathcal{O}(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)} - D)) \simeq \{0\}.$$

Since the Chern character $ch(\widetilde{V}_m)$ of $\widetilde{V}_m$ is $(2m)^g$, from the above vanishing theorem and the theorem of Riemann-Roch-Hirzebruch we have

Theorem 2.6. *If $k \geq g+2$, then*

$$\dim J_{k,m}^{cusp}(\Gamma_g(4mN)) = \dim S_{k,m}(\Gamma_g(4mN)) = (2m)^g \dim S_{k-1/2}(\Gamma_g(4mN)),$$

where $S_{k-1/2}(\Gamma_g(4mN))$ is the space of Siegel cusp forms of weight $k-1/2$.

$M \in \Gamma_g$ acts on $S_{k,m}(\Gamma_g(4mN))$ as follows:

$$MF(M\langle Z\rangle) = \det(CZ+D)^{k-1/2}\,\overline{u(M)}\,F(Z).$$

Since $\Gamma_g(4mN)$ acts trivially, $\Gamma_g/\Gamma_g(4mN)$ acts on $S_{k,m}(\Gamma_g(4mN))$. Hence the dimension of $J_{k,m}^{cusp}(\Gamma_g) \simeq S_{k,m}(\Gamma_g)$ is calculated as an invariant subsapace of $S_{k,m}(\Gamma_g(4mN))$ by using the holomorphic Lefschetz fixed point formula ([AS]).

We recall the holomorphic Lefschetz fixed point formula. Let X be a compact complex manifold and V a holomorphic vector bundle of rank n on X, and let G be a finite group of automorphism of the pair (X, V). For $g \in G$ let X^g be the set of fixed points of g. X^g is a disjoint union of submanifolds of X. Let

$$X^g = \sum_\alpha X^g_\alpha$$

be the irreducible decomposition of X^g, and let

$$N^g_\alpha = \sum_\theta N^g_\alpha(\theta)$$

denote the normal bundle of X^g_α decomposed according to the eigenvalues $e^{i\theta}$ of g. We put

$$\mathcal{U}^\theta(N^g_\alpha(\theta)) = \prod_\beta \left(\frac{1 - e^{-x_\beta - i\theta}}{1 - e^{-i\theta}}\right)^{-1},$$

where the Chern class of $N^g_\alpha(\theta)$ is

$$c(N^g_\alpha(\theta)) = \prod_\beta (1 + x_\beta).$$

Let $\mathcal{T}(X^g_\alpha)$ be the Todd class of X^g_α. Let $V|X^g_\alpha$ be the restriction of V to X^g_α and $ch(V|X^g_\alpha)(g)$ the Chern character of $V|X^g_\alpha$ with g-action (see below). Put

$$\tau(g, X^g_\alpha) = \left\{\frac{ch(V|X^g_\alpha)(g) \cdot \prod_\theta \mathcal{U}^\theta(N^g_\alpha(\theta)) \cdot \mathcal{T}(X^g_\alpha)}{\det(1 - g|(N^g_\alpha)^*)}\right\}[X^g_\alpha]$$

and

$$\tau(g) = \sum_\alpha \tau(g, X^g_\alpha).$$

Note that in the definition of $\tau(g, X^g_\alpha)$ the terms except $ch(V|X^g_\alpha)(g)$ depend only on the base space X^g_α. We have

Theorem 2.7. ([AS])

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(g \mid H^i(X, \mathcal{O}(V))) = \tau(g).$$

To use the Lefschetz fixed point formula we have to classify the fixed points (sets). We classify (the irreducible components of) the fixed points of G in the following sense. Let Φ_1 and Φ_2 be the fixed points (sets). Φ_1 and Φ_2 is called *equivalent* if there is an element of G which maps Φ_1 biholomorphically to Φ_2. Let Φ be one of fixed points (sets) and let

$$C(\Phi) = \{g \in G \mid g(x) = x \text{ for any } x \in \Phi\}.$$

Let $g \in C(\Phi)$ and $H = \langle g \rangle$ the subgroup of $C(\Phi)$ which is generated by g and let $\widehat{H}$ be the character group of H. Let $\chi \in \widehat{H}$ and let V_χ be the subbundle of V consisting of the vectors

$v \in V|\Phi$ such that $g(v) = \chi(g)v$ for any $g \in C(\Phi)$. $V|\Phi$ is a direct sum of subbundles V_χ's. We define

$$ch(V|\Phi)(g) := \sum_\chi \chi(g)\, ch(V_\chi).$$

Now we return to our case. The fixed points (sets) in the quotient space $X_2(4mN)$ were classified in [G]. The fixed points in the divisor at infinity are classified easily. In total there are 25 kinds of fixed points (sets). Let Φ be one of these fixed points (sets). We can prove

Lemma 2.8. *The direct summands $\widetilde{V}_\chi$ of $\widetilde{V}_m|\Phi$ are also flat vector bundles.*

Hence for $M \in C(\Phi)$ we have

$$ch(\widetilde{V}_m|\Phi)(M) = \sum_\chi \chi(M)\, ch(\widetilde{V}_\chi) = \sum_\chi \chi(M) \operatorname{rank} \widetilde{V}_\chi = \operatorname{Tr} \overline{u(M)}$$

and

$$ch(\widetilde{V}_m \otimes \widetilde{H}_g^{\otimes(2k-1)} \otimes [D]^{\otimes(-1)}|\Phi)(M) = \operatorname{Tr} \overline{u(M)}\, ch(\widetilde{H}_g^{\otimes(2k-1)} \otimes [D]^{\otimes(-1)}|\Phi)(M).$$

Therefore we can apply the data when we computed the dimension of $S_k(\Gamma_2)$ by using the holomorphic Lefschetz fixed point formula ([Ts1]) and what we have to do is

(a) to determine $\det(CZ+D)^{1/2}\, u(M)$ for $M \in C(\Phi)$,

(b) to evaluate the Gaussian sums which appear in $\operatorname{Tr} \overline{u(M)}$,

and

(c) to execute a terrible computation.

§3. The case $\mathbf{m} = 1$

The case of index one is very important concerning Saito-Kurokawa lifting.

Proposition 3.1.

$$\begin{aligned}\sum_{k=0}^{\infty} \dim J_{k,1}^{cusp}(\Gamma_2)\, t^k &= \sum_{k=0}^{\infty} \texttt{JacobiTwo[k,1]}\, t^k + t^3 \\ &= \frac{t^{10} + t^{12} + t^{14} + 2t^{16} + t^{18} + t^{21} - t^{26} + t^{27} - t^{28} + t^{29} + t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.\end{aligned}$$

Proof. Let φ_4 be the Eisenstein series of degree 2 and weight 4. Then if $f \in J_{k,1}^{cusp}(\Gamma_2)$, $\varphi_4 f \in J_{k+4,1}^{cusp}(\Gamma_2)$. Since $\dim J_{k,1}^{cusp}(\Gamma_2) = \texttt{JacobiTwo[k,1]} = 0$ for $k = 4, 5, 6, 7$, we have $\dim J_{k,1}^{cusp}(\Gamma_2) = 0$ for $k = 0, 1, 2, 3$. On the other hand we have $\texttt{JacobiTwo[k,1]} = 0$ for $k = 0$, 1, 2 and $\texttt{JacobiTwo[3,1]} = -1$. Hence the equality of the first line holds. □

We use the following theorem concerning the surjectivity of Φ-operator ([D]).

Theorem 3.2. *If k is even and $k \geq 6$, then*

$$\dim J_{k,m}(\Gamma_2) = \dim J_{k,m}^{cusp}(\Gamma_2) + d_m^1 \dim J_{k,m}^{cusp}(\Gamma_1) + d_m^0 \dim J_{k,m}^{cusp}(\Gamma_0),$$

where d_m^1, $d_m^0 \geq 1$. We define that $J_{k,m}^{cusp}(\Gamma_0) = \mathbf{C}$.

Remark 3.3. If m is square free, then $d_m^1 = d_m^0 = 1$.

From above results we have

Corollary 3.4.

$$\sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_2)\, t^k = \frac{t^4 + t^6 + t^{10} + t^{12} + t^{21} + t^{27} + t^{29} + t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}.$$

Proof. Let $M_k(\Gamma_g)$ be the space of Siegel modular forms of weight k with respect to Γ_g. We have $\dim M_4(\Gamma_3) = 1$ ([Ty]). Let α_4 the base of $M_4(\Gamma_3)$ and let $f_{4,1}$ be the coefficient of $\mathbf{e}(Z_{33})$ in the Fourier-Jacobi expansion of α_4. Since $f_{4,1}$ is not identically zero, we have $\dim J_{4,1}(\Gamma_2) \geq 1$. On the other hand we have $\dim J_{4,1}(\Gamma_2) \leq \dim J_{4,1}^{cusp}(\Gamma_2) + \dim J_{4,1}(\Gamma_1) = 0 + 1 = 1$. Hence we have $\dim J_{4,1}(\Gamma_2) = 1$ and Φ-operator is also surjective in case $k = 4$. Since $\sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_1)\, t^k = \frac{t^4 + t^6}{(1-t^4)(1-t^6)}$, we have

$$\sum_{k=0}^{\infty} \dim J_{k,1}(\Gamma_2)\, t^k = \sum_{k=0}^{\infty} \dim J_{k,1}^{cusp}(\Gamma_2)\, t^k + \frac{t^4 + t^6}{(1-t^4)(1-t^6)}.$$

□

Remark 3.5. $\bigoplus_{k=0}^{\infty} J_{k,1}(\Gamma_2)$ is a $\bigoplus_{k=0}^{\infty} M_k(\Gamma_2)$-module. Because we have

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_2)\, t^k = \frac{1 + t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

([Ig1]), $\bigoplus_{k=0}^{\infty} J_{k,1}(\Gamma_2)$ does not have a nice structure as a $\bigoplus_{k=0}^{\infty} M_k(\Gamma_2)$-module but will be a free $\bigoplus_{k=0}^{\infty} M_{2k}(\Gamma_2)$-module of rank 8.

Remark 3.6. Let $M_{2k-1/2}^{+}(\Gamma_0^2(4))$ be the plus space in $M_{2k-1/2}(\Gamma_0^2(4))$. Then there is an isomorphism between $J_{2k,1}(\Gamma_2)$ and $M_{2k-1/2}^{+}(\Gamma_0^2(4))$ ([Kh], [Ib1]). The dimension of $M_{k-1/2}(\Gamma_0^2(4))$ was computed in [Ts3] and the structure of $\bigoplus_{k=0}^{\infty} M_{k-1/2}(\Gamma_0^2(4))$ was determined in [Ib2]. Hence if one finds the generators of $\bigoplus_{k=0}^{\infty} M_{2k-1/2}^{+}(\Gamma_0^2(4))$, he can find the generators of $\bigoplus_{k=0}^{\infty} J_{2k,1}(\Gamma_2)$. According to T. Ibukiyama, he found the generators of $\bigoplus_{k=0}^{\infty} M_{2k-1/2}^{+}(\Gamma_0^2(4))$.

APPENDIX

Since we explained nothing about the detailed computation, we show the computation of the case of degree one here. Of course our result coincides with the result of Eichler-Zagier ([EZ] p.105 and p.121). In the computation we use Lemma A.2. We can also compute the dimension of $J_{k,m}^{cusp}(\Gamma)$ for any congruence subgroup Γ of $SL(2,\mathbf{Z})$ (cf. [Ts2] §1).

Theorem A.1. *If $k \geq 3$, then the dimension of the space of Jacobi cusp forms $J_{k,m}^{cusp}(\Gamma_1)$ is given by the following Mathematica function:*

```
JacobiOne[k_,m_]:=Block[{a,r,p,lk,x,y},
  mod[x_,y_]:=Mod[x,y]+1;
  r=0;
  While[EvenQ[m/2^r],r++;];
  p=m/2^r;
  a=m*(2*k-15)/2^3/3+Sum[Mod[x^2,4*m],{x,1,2*m-1}]/m/2^3;
  a=a+If[Mod[k,2]==0,1,-1]*((2*k-15)/2^3/3+Mod[m,4]/2^3);
  lk={1,-1,-1,1};
  a=a+If[Mod[m,2]==0,1,0]*lk[[mod[k,4]]]/2^2;
  lk={0,-1,-1,0,1,1};
  a=a+lk[[mod[k,6]]]/2/3;
  If[Mod[m,3]==0,Goto[label1],0];
  lk={0,-1,1};
  a=a+If[Mod[p,3]==1,1,-1]*If[Mod[r,2]==0,-1,1]*lk[[mod[k,3]]]/2/3;
  Goto[label2];
  Label[label1];
  lk={2,-1,-1};
  a=a+lk[[mod[k,3]]]/2/3;
  Label[label2];
  Return[a];
  ]
```

Proof. Let $M \in SL(2,\mathbf{Z})$ and let $u(M) = (u_{rs}(M))_{r,s}$. We denote $u_{rs}(M)$ by u_{ab}, where $r = \dfrac{a}{2m}$ and $s = \dfrac{b}{2m}$.

(a) Let $M = 1_2$. We choose $(cz+d)^{1/2} = 1^{1/2} = 1$. Then $u(M) = 1_{2m}$. Hence $\mathrm{Tr}\,(\overline{u(M)}) = 2m$. So we have

$$\begin{aligned}\tau(1_2, \overline{X}_1(4mN)) &= (2m)\dim S_{k-1/2}(\Gamma_1(4mN)) \\ &= 2m\left(\left(k-\frac{3}{2}\right)\frac{(4mN)^3}{24} - \frac{(4mN)^2}{4}\right)\prod(1-p^{-2}),\end{aligned}$$

where p is over the prime numbers which divide $4mN$. Therefore the contribution of 1_2 to the dimension of $J_{k,m}^{cusp}(\Gamma_1)$ is

$$\frac{\tau(1_2, \overline{X}_1(4mN))}{[\Gamma_1 : \Gamma_1(4mN)]} = 2m\left(\left(k-\frac{3}{2}\right)\frac{1}{24} - \frac{1}{16mN}\right).$$

(b) Let $M_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ $(1 \le r \le 4mN-1)$ and let $\zeta = \mathbf{e}(1/4mN)$. M_r fixes the cusp P_∞ of $\overline{X}_1(4mN)$. We choose $(cz+d)^{1/2} = 1^{1/2} = 1$. Then we have $u_{aa} = \mathbf{e}(a^2r/4m)$ and $u_{ab} = 0$, otherwise. Hence

$$\begin{aligned}
\tau(M_r, P_\infty) &= \frac{ch(\widetilde{V}_m \otimes \overline{H}_1^{\otimes(2k-1)} \otimes [D]^{\otimes(-1)}|P_\infty)(M_r)}{(1 - M_r|(N_\alpha^{M_r})^*)} \\
&= \frac{\mathrm{Tr}\,\overline{u(M_r)}\, ch(\overline{H}_1^{\otimes(2k-1)} \otimes [D]^{\otimes(-1)}|P_\infty)(M_r)}{(1-\zeta^{-r})} \\
&= \sum_{a=0}^{2m-1} \mathbf{e}(-a^2r/4m)\frac{\zeta^{-r}}{(1-\zeta^{-r})}.
\end{aligned}$$

For an integer a, we denote by Mod$[a, 4mN]$ the integer such that $a \equiv \mathrm{Mod}[a, 4mN] \pmod{4mN}$ and $0 \le \mathrm{Mod}[a, 4mN] < 4mN$. Then similarly as [Ts1] Example (5.4), the sum of the contributions of M_r $(1 \le r \le 4mN-1)$ is equal to

$$\begin{aligned}
\frac{1}{|C(P_\infty)|}\sum_{r=1}^{4mN-1}\sum_{a=0}^{2m-1}\frac{\mathbf{e}(-a^2r/4m)}{(\zeta^r-1)} &= \frac{1}{8mN}\sum_{a=0}^{2m-1}\sum_{r=1}^{4mN-1}\frac{\zeta^{-a^2rN}}{(\zeta^r-1)} \\
&= \frac{1}{8mN}\sum_{a=0}^{2m-1}\left(\frac{1-4mN}{2} + \mathrm{Mod}[a^2, 4m]\cdot N\right) \\
&= -\frac{m}{2} + \frac{1}{8N} + \frac{1}{8m}\sum_{a=1}^{2m-1}\mathrm{Mod}[a^2, 4m].
\end{aligned}$$

(c) Let $M = -1_2$. We choose $(cz+d)^{1/2} = (-1)^{1/2} = i$. Then $u_{-aa} = -i$ and $u_{ab} = 0$, otherwise. Hence $\mathrm{Tr}\,(\overline{u(M)}) = \overline{u_{00} + u_{mm}} = 2i$. Therefore the contribution of -1_2 is

$$\frac{\tau(-1_2, \overline{X}_1(4mN))}{[\Gamma_1 : \Gamma_1(4mN)]} = 2(-1)^k\left(\left(k-\frac{3}{2}\right)\frac{1}{24} - \frac{1}{16mN}\right).$$

(d) Let $M = -M_r$. We choose $(cz+d)^{1/2} = (-1)^{1/2} = i$. Then $u_{-aa} = -i\cdot\mathbf{e}(a^2r/4m)$ and $u_{ab} = 0$, otherwise. Hence $\mathrm{Tr}\,(\overline{u(M)}) = \overline{u_{00} + u_{mm}} = i\cdot(1 + \mathbf{e}(-m^2r/4m))$. Therefore the sum of the contributions of $-M_r$ $(1 \le r \le 4mN-1)$ is equal to

$$\begin{aligned}
\frac{(-1)^k}{8mN}\sum_{r=1}^{4mN-1}\frac{1+\zeta^{-m^2rN}}{(\zeta^r-1)} &= \frac{(-1)^k}{8mN}\left((1-4mN) + \mathrm{Mod}[m^2, 4m]\cdot N\right) \\
&= (-1)^k\left(-\frac{1}{2} + \frac{1}{8mN} + \frac{\mathrm{Mod}[m, 4]}{8}\right).
\end{aligned}$$

(e) Let $M_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. M_i fixes $i = \sqrt{-1}$. We choose $(cz+d)^{1/2} = i^{1/2} = (1+i)/\sqrt{2}$. Then $u_{ab} = (1-i)\mathbf{e}(-ab/2m)/2\sqrt{m}$. Therefore the contribution of M_i is

$$\frac{1}{|C(i)|}\frac{1+i}{2\sqrt{m}}\sum_{a=0}^{2m-1}\mathbf{e}(a^2/2m)\frac{ch(\overline{H}_1^{\otimes(2k-1)}|\,i)(M_i)}{(1-M_i|(N_\alpha^{M_i})^*)} = \frac{1+i}{8\sqrt{m}}\frac{\left((1+i)/\sqrt{2}\right)^{2k-1}}{1+1}\sum_{a=0}^{2m-1}\mathbf{e}(a^2/2m)$$
$$= \begin{cases} \dfrac{(i)^k(1+i)}{8}, & \textit{if } m \textit{ is even,} \\ 0, & \textit{if } m \textit{ is odd.} \end{cases}$$

The contribution of M_i^{-1} is the complex conjugate of the contribution of M_i.

(f) Let $\rho = \mathbf{e}(1/3)$ and $M_\rho = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. M_ρ fixes ρ. We choose $(cz+d)^{1/2} = (-\rho)^{1/2} = i\rho^2$. Then $u_{ab} = (1+i)\mathbf{e}(a(2b-a)/4m)/2\sqrt{m}$. Therefore the contribution of M_ρ is

$$\frac{1}{|C(\rho)|}\frac{1-i}{2\sqrt{m}}\sum_{a=0}^{2m-1}\mathbf{e}(-a^2/4m)\frac{ch(\overline{H}_1^{\otimes(2k-1)}|\,\rho)(M_\rho)}{(1-M_\rho|(N_\alpha^{M_\rho})^*)} = \frac{1-i}{12\sqrt{m}}\frac{(i\rho^2)^{2k-1}}{1-\rho^2}\sum_{a=0}^{2m-1}\mathbf{e}(-a^2/4m)$$
$$= \frac{(-\rho)^{k+1}}{6(1-\rho^2)}.$$

The contribution of M_ρ^{-1} is the complex conjugate of the contribution of M_ρ.

(g) Let $M = M_\rho^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. M fixes ρ. We choose $(cz+d)^{1/2} = (\rho^2)^{1/2} = \rho$. Then $u_{ab} = (-1-i)\mathbf{e}(b(b+2a)/4m)/2\sqrt{m}$. Therefore the contribution of M is

$$\frac{1}{|C(\rho)|}\frac{-1+i}{2\sqrt{m}}\sum_{a=0}^{2m-1}\mathbf{e}(-3a^2/4m)\frac{ch(\overline{H}_1^{\otimes(2k-1)}|\,\rho)(M)}{(1-M|(N_\alpha^{M})^*)} = \frac{-1+i}{12\sqrt{m}}\frac{\rho^{2k-1}}{1-\rho}\sum_{a=0}^{2m-1}\mathbf{e}(-3a^2/4m)$$
$$= \begin{cases} \dfrac{(\rho^2)^k}{6}, & \textit{if } 3 \mid m, \\ \left(\dfrac{p}{3}\right)\cdot\dfrac{(\rho^2)^{k+1}(-1)^{r+1}}{6(1-\rho)}, & \textit{if } 3 \nmid m, \end{cases}$$

where $m = 2^r p$ and p is an odd integer. $\left(\frac{p}{3}\right)$ means the Legendre symbol. The contribution of M^{-1} is the complex conjugate of the contribution of M. □

Lemma A.2. *Let $\zeta = \mathbf{e}(1/4mN)$ and let $m = 2^r p$, where p is an odd integer. Then we have*

(1)
$$\sum_{r=1}^{4mN-1}\frac{\zeta^{-ar}}{(1-\zeta^r)} = \frac{4mN-1}{2} - \mathrm{Mod}[a, 4mN].$$

(2)
$$\sum_{a=0}^{2m-1}\mathbf{e}(a^2/2m) = \begin{cases} (1+i)\sqrt{2m}, & \textit{if } m \textit{ is even,} \\ 0, & \textit{if } m \textit{ is odd.} \end{cases}$$

(3)
$$\sum_{a=0}^{2m-1}\mathbf{e}(3a^2/4m) = \begin{cases} (1+i)\sqrt{3m}, & \textit{if } 3 \mid m, \\ \left(\frac{p}{3}\right)(-1)^{r+1}(-1+i)\sqrt{m}, & \textit{if } 3 \nmid m. \end{cases}$$

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